DTIC FILE COPY

ADE 951163

②

AD-A199 191

TECHNICAL REPORT RD-GC-86-16

VIBRATION OF PRESSURIZED CYLINDERS

Carl H. Warren
Guidance and Control Directorate
Research, Development, and Engineering Center
and
A. R. Barbin
Auburn University
Auburn, Alabama

JANUARY 1988

U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35898-5000

Approved for public release; Distribution unlimited.

SMI FORM 1021, 1 AUG 85 PREVIOUS EDITION IS OBSOLETE

88 10 4 070

DISPOSITION INSTRUCTIONS

DESTROY THIS REPORT WHEN IT IS NO LONGER NEEDED. DO NOT RETURN IT TO THE ORIGINATOR.

DISCLAIMER

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

TRADE NAMES

USE OF TRADE NAMES OR MANUFACTURERS IN THIS REPORT DOES NOT CONSTITUTE AN OFFICIAL INDORSEMENT OR APPROVAL OF THE USE OF SUCH COMMERCIAL HARDWARE OR SOFTWARE.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

AD-A199 191

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188
Exp Date Jun 30, 1986

1a REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

1b RESTRICTIVE MARKINGS

2a SECURITY CLASSIFICATION AUTHORITY

2b DECLASSIFICATION/DOWNGRADING SCHEDULE

3 DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release; distribution unlimited.

4 PERFORMING ORGANIZATION REPORT NUMBER(S)
RD-GC-86-16

5 MONITORING ORGANIZATION REPORT NUMBER(S)

6a NAME OF PERFORMING ORGANIZATION
US Army Missile Command

6b OFFICE SYMBOL (If applicable)
AMSMI-RD-GC

7a NAME OF MONITORING ORGANIZATION

6c. ADDRESS (City, State, and ZIP Code)
Redstone Arsenal, AL 35898-5254

7b ADDRESS (City, State, and ZIP Code)

8a. NAME OF FUNDING/SPONSORING ORGANIZATION

8b OFFICE SYMBOL (If applicable)

9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

8c. ADDRESS (City, State, and ZIP Code)

10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO
72W801	3R	00	7260

11 TITLE (Include Security Classification)
Vibration of Pressurized Cylinders

12 PERSONAL AUTHOR(S)
Carl H. Warren, Guidance and Control Directorate, and A. R. Barbin, Auburn University

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM Jan 83 TO Jul 86	1988 January	20

16 SUPPLEMENTARY NOTATION

17 COSATI CODES

FIELD	GROUP	SUB-GROUP

18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

A study was conducted of the effects of internal pressure and media on the vibration of small cylindrical pressure vessels. Utilizing approximate expressions for elastic potential and kinetic energies of the cylinder-fluid system, LaGrange's equation was applied and an expression developed for predicting the frequency of vibration as a function of the pressure and density of the fluid for axial wave factors $\lambda = m\pi R/L < 0.60$, thickness ratios $\delta/R > 0.03$ and number of circumferential waves $n \geq 2$. End conditions are empirically included in the measured zero-pressure frequency ω_0. A series of experiments on a small cylinder of the type used as pneumatic power sources on missiles confirmed the predictions over a range of pressures from 0-34 MPa, with helium and nitrogen gases. A technique of nondestructive pressure measurements is proposed based on the results of this study.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT
☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS

21 ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED

22a NAME OF RESPONSIBLE INDIVIDUAL
Carl H. Warren

22b TELEPHONE (Include Area Code)
AV 746-1225/205-876-1225

22c OFFICE SYMBOL
AMSMI-RD-GC

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

TABLE OF CONTENTS

		Page
I.	INTRODUCTION	1
II.	ANALYSIS	2
III.	EXPERIMENTS	6
IV.	RESULTS-DISCUSSION	8
V.	SUMMARY	10
REFERENCES		11
GLOSSARY		12
APPENDIX	KINETIC ENERGY OF A FLUID IN A VIBRATING CIRCULAR CYLINDER	A-1

I. INTRODUCTION

The essential features of free vibration of unpressurized cylindrical shells have been well established by previous investigations of infinitely long [1] and finite-length [2,3] cylinders. Cylindrical shells can vibrate with various combinations of axial and circumferential waves. For any given wave pattern, three distinct natural frequencies are possible depending on the dominant amplitude (radial, axial, or circumferential) of the motion [1].

According to Arnold and Warburton [1], investigations of cylindrical shells can proceed either directly to a set of differential equations via equilibrium considerations of a small element or indirectly by assuming vibration forms compatible with the given end conditions, deriving expressions for strain and kinetic energies, and using the Lagrange dynamical equations to arrive at the differential equations. Discrepancies between the results of various investigations can be traced to different approximations utilized in relating strains to the extensional strains of the middle surface and its curvature changes; for an accounting of the relations used by various authors, see the review presented in Reference [3].

Previous investigations of cylindrical shell vibrations have been limited to unpressurized vessels. The purpose of this study was to assess the effect of pressurization on the natural frequency. The analysis is based on a simplified energy method using the assumption that radial displacements dominate. The zero-pressure natural frequency is assumed to be given either by existing predictive techniques or measurements.

II. ANALYSIS

Figure 1 illustrates the cylinder considered in this treatment. The ends of the cylinder $z = 0$ and $z = L$ are assumed to remain circular during vibration, and the middle surface is presumed to move in a strictly radial direction; thus, the displacements of the middle surface are

$$u = U \cos (n\theta) \sin (m\pi z/L) \tag{1}$$

$$v = w = 0$$

where n is the number of circumferential waves and m is the number of axial halfwaves; U is periodic in time

$$U = U_0 \sin \omega t \quad . \tag{2}$$

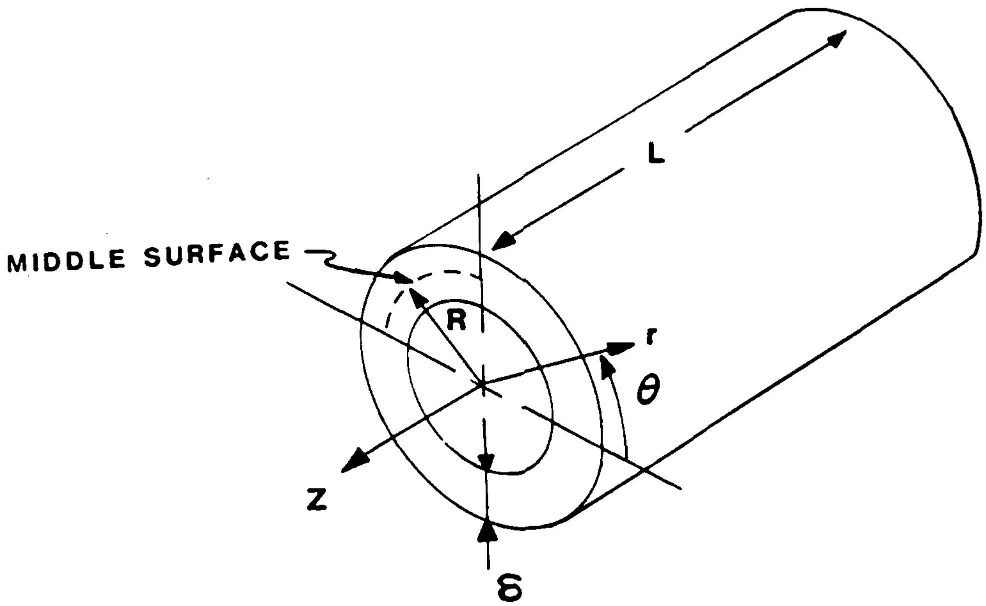


Figure 1. Cylinder.

The total potential energy of the cylinder-fluid system is the sum of the cylinder bending and stretching energies and the fluid expansion energy. In computing the energy of cylinder stretching, it is assumed that tension is due to internal pressure only, and per unit length is pR in the circumferential direction and pR/2 in the axial direction; fluid pressure p is assumed constant. Upon deformation the length of the middle surface increases by

$$\Delta\ell_\theta = \int_0^{2\pi} \left([(R+u)^2 + (\partial u/\partial\theta)^2]^{1/2} - R\right)d\theta \tag{3}$$

in the θ-direction. The integrand of this expression can be expanded and, upon retention of the first-order terms only (for small displacements $u/R \ll 1$ and slopes $\partial u/R\partial\theta \ll 1$), approximated by

$$\Delta\ell_\theta = \int_0^{2\pi} [u + \frac{1}{2R}(\frac{\partial u}{\partial\theta})^2]d\theta \quad . \tag{4}$$

Similar treatment of extension parallel to the z-axis yields

$$\Delta\ell_z = \int_0^{2\pi} \frac{1}{2}(\partial u/\partial z)^2\ dz \tag{5}$$

for extension of a generator of the cylinder. Equations (4) and (5) can be used to form the epxression for stretching energy

$$V_S = \int_0^{2\pi}\int_0^L p[u + \frac{1}{2R}(\frac{\partial u}{\partial\theta})^2]Rd\theta dz + \int_0^{2\pi}\int_0^L \frac{pR^2}{4}(\frac{\partial u}{\partial z})^2\ d\theta dz \quad . \tag{6}$$

The volume of fluid inclosed by the cylinder increases during deformation by the amount

$$\int_0^{2\pi}\int_0^L uRd\theta dz, \tag{7}$$

thus, the fluid expansion energy is given by

$$V_E = -\int_0^{2\pi}\int_0^L puRd\theta dz \quad . \tag{8}$$

Equations (6) and (8) can be combined to yield the sum of the stretching and expanison energies as

$$V_S + V_E = \int_0^{2\pi}\int_0^L p[\frac{1}{2R}(\frac{\partial u}{\partial\theta})^2 + \frac{R}{4}(\frac{\partial u}{\partial z})^2]Rd\theta dz \tag{9}$$

or, the displacement of equation (1) can be used

$$V_S + V_E = \pi Ln^2U^2p[1 + (\lambda/n)^2]/4 \tag{10}$$

where

$$\lambda \equiv m\pi R/L \quad .$$

The potential energy of cylinder bending V_B is calculated from the expression

$$V_B = \int_0^{2\pi} \int_0^{L} \int_{-\delta/2}^{\delta/2} \frac{E}{2(1-\sigma^2)} [e_z{}^2 + e_\theta{}^2 + 2\sigma e_z e_\theta + (\frac{1-\sigma}{2}) \gamma_{\theta z}{}^2] R d\theta dz dx \qquad (11)$$

where, following Love [5], it has been assumed that normal stress in the radial direction, σ_{rr}, is small and the shear strains $\gamma_{r\theta}$ and γ_{rz} are zero. The strains e_z and e_θ are presumed to be linear functions of x, with zero midplane strain; thus [5]

$$e_z = -x(\partial^2 u/\partial x^2) = (m\pi/L)^2 \, Ux \cos(n\theta) \sin(m\pi z/L) \qquad (12)$$

$$e_\theta = -x(\partial^2 u/\partial \theta^2)/R^2 = (n/R)^2 \, Ux \cos(n\theta) \sin(m\pi z/L)$$

where the displacement Equation (1) has been invoked. These strain expressions may be substituted into Equation (11) resulting in the expression

$$V_B = \frac{\pi E L U^2}{4(1-\sigma^2)} \frac{1}{12} (\frac{\delta}{R})^3 (\lambda^2+n^2)^2 \quad . \qquad (13)$$

The total kinetic energy of the cylinder-fluid system is the sum of the kinetic energy of the cylinder

$$T_c = \int_0^{2\pi} \int_0^{L} \int_{R-\delta/2}^{R+\delta/2} \frac{1}{2} \rho(\frac{\partial u}{\partial t})^2 \, r d\theta dz dr = \frac{\pi}{4} \rho R L (\dot{U})^2 \delta \qquad (14)$$

and the kinetic energy of the fluid (Equation (A-7). Thus,

$$T = T_c + T_f = \pi L R \rho_e (\dot{U})^2 \delta/4 \qquad (15)$$

where ρ_e is an "equivalent density" defined by

$$\rho_e \equiv \rho + R\rho_f/(n\delta) \quad . \qquad (16)$$

Equations (10) and (13) may be added to yield the total potential energy of the cylinder-fluid system; thus,

$$V = \frac{\pi E L U^2}{4(1-\sigma^2)} \frac{\delta^3}{12 R^3} (\lambda^2+n^2)^2 + \frac{\pi}{4} L n^2 U^2 p[1+(\lambda/n)^2] \qquad (17)$$

Equations (17) and (15) may then be utilized in Lagrange's equation

$$\frac{d}{dt} (\partial T/\partial \dot{U}) - \partial T/\partial U = -\partial V/\partial U \qquad (18)$$

to yield the expression

$$\frac{R\delta\rho_e}{2}\ddot{U} = -U\left\{\frac{E}{24}\left(\frac{\delta}{R}\right)^3\frac{(\lambda^2+n^2)^2}{(1-\sigma^2)} + \frac{p}{2}(\lambda^2+n^2)\right\} \quad . \tag{19}$$

For periodic oscillations, Equation (2), this expression reduces to

$$\omega^2 R\delta\rho_e = \frac{E}{12(1-\sigma^2)}\left(\frac{\delta}{R}\right)^3(\lambda^2+n^2)^2 + p(\lambda^2+n^2) \tag{20}$$

At zero gage pressure, $p = 0$, $\rho_e \simeq \rho$, and Equation (20) reduces, in that instance, to

$$\omega_o^2 R\delta\rho = \frac{E}{12(1-\sigma^2)}\left(\frac{\delta}{R}\right)^3(\lambda^2+n^2)^2 \tag{21}$$

where ω_o denotes the unpressurized oscillation frequency. Equation (20) can be conveniently nondimensionalized by division by Equation (21); after some manipulation the results are

$$\omega^2 = \omega_o^2\frac{\rho}{\rho_e}\left[1 + \frac{12}{n^2}\frac{p}{E}\left(\frac{R}{\delta}\right)^3\frac{(1-\sigma^2)}{(\lambda/n)^2+1}\right] . \tag{22}$$

This expression can be simplified, with the aid of Equation (16) to the form

$$\omega = \omega_o\left\{\frac{1}{1+\frac{1}{n}\frac{R}{\delta}\frac{\rho_f}{\rho}}\left[1 + \frac{2}{n^2}\frac{p}{E}\left(\frac{R}{\delta}\right)^3\frac{(1-\sigma^2)}{(m\pi R/nL)^2+1}\right]\right\}^{1/2} \tag{23}$$

The derivation of Equation (23) is based on the assumption that the numbers of axial and circumferential waves do not depend on pressure. The zero-pressure frequency ω_o is considered to be given either by experiment or some predictive technique other than Equation (21). Equation (23) indicates that cylinder pressurization tends to increase the frequency through the increased strain energy with a moderating influence of the kinetic energy of the fluid.

III. EXPERIMENTS

A series of oscillation frequency measurements were made on a small gas storage bottle, typical of those used as pneumatic storage supplies on certain missiles [4]. Figure 2 shows the bottle and the dimensions of the bottle. Qualitative preliminary measurements were conducted to determine the numbers of axial and circumferential waves existing during free vibration; View A of Figure 3 shows the test setup used. An oscillator was used to drive a small loudspeaker placed adjacent to the bottle and adjusted to a frequency which produced a peak output of an accelerometer glued to the bottle. By moving the accelerometer axially and circumferentially, and by comparing its output to a second accelerometer fixed to the bottle, the node lines were determined. Two circumferential waves ($n = 2$) and one axial half-wave ($m = 1$) were found to exist for pressures from atmospheric up to 34.5 MPa (5000 psig) for nitrogen and helium gases. For the bottle tested, $m\pi R/L \simeq 0.576$ and $\delta/R = 0.0867$. According to Arnold and Warburton [1] these ratios indicate that the minimum frequency should occur for $n = 2$, as observed.

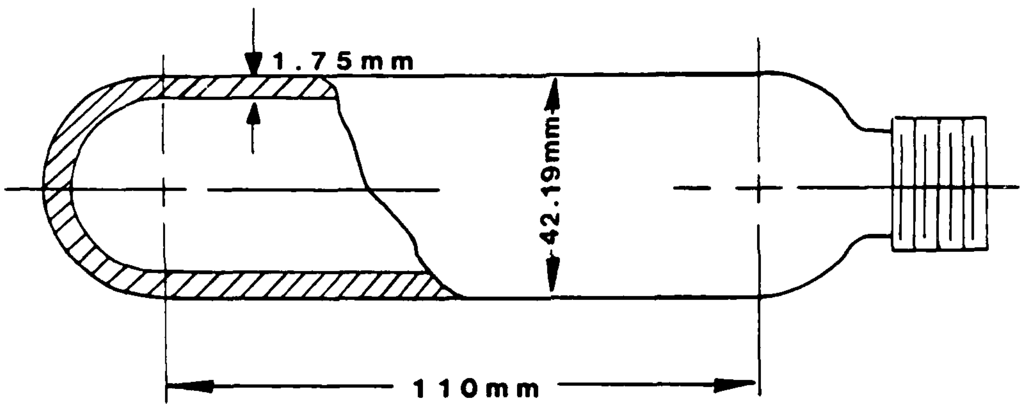
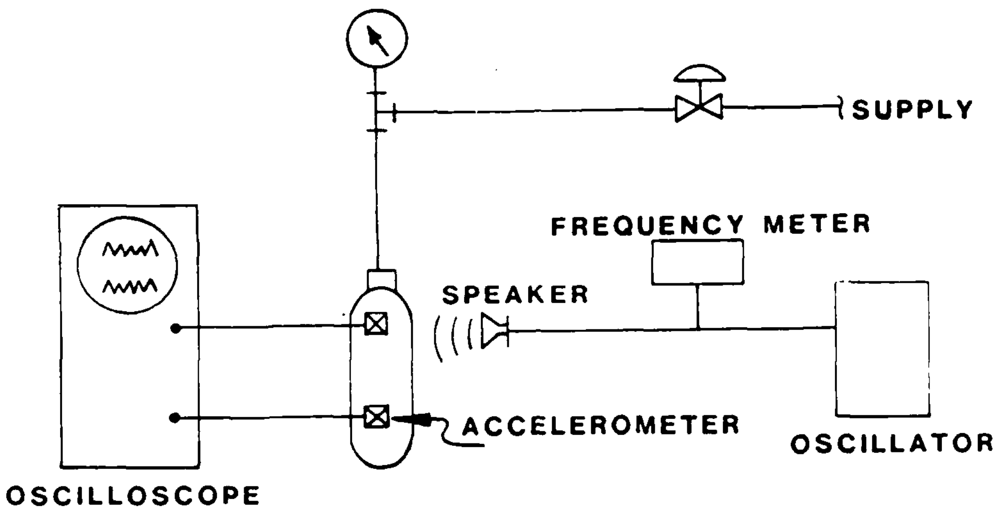


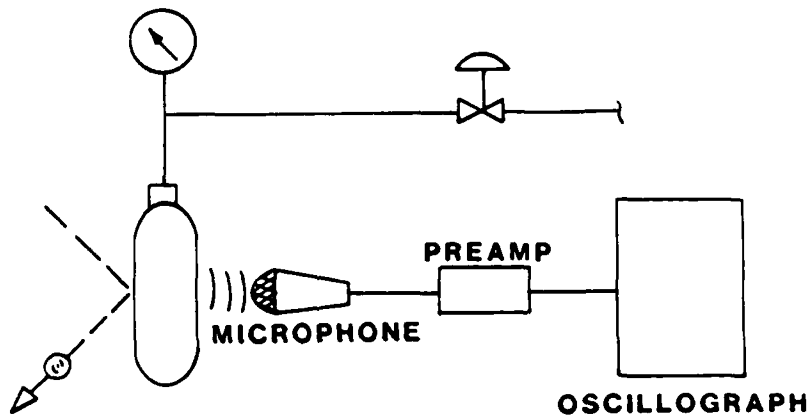
Figure 2. Gas storage bottle.

Frequency measurements were conducted as shown in the test setup in View B of Figure 3. Vibration of the bottle was initiated by sharply tapping it with a small hammer. The output of a microphone placed adjacent to the bottle was recorded on an oscillograph. The measurements were completed by manually counting the cycles in a specified time, short enough (0.100 s) to avoid signal dropout due to damping. Tests were made for nitrogen and helium gases over a range of pressures from 0-34.5 MPa gage. Pressurization between measurements caused significant temperature changes. Great care was taken to cool the cylinder down to room temperature (20 °C) via a water bath prior to each measurement.

Measurement uncertainties are estimated as two parts in two hundred for frequency (~ 50 Hz) and 0.5 MPa for pressure. The dimensions of the storage bottle shown in Figure 2 have an uncertainty of 0.025 mm.



VIEW A. **Determination of Nodes**



VIEW B. **Frequency Measurements**

Figure 3. Experimental setup.

IV. RESULTS-DISCUSSION

The results of the frequency measurements are presented in Figure 4 for both helium and nitrogen. The zero-pressure frequency was 3550 Hz for each. At the highest pressures for which data was taken ($\sim$ 34 MPa), the frequency for helium increased by approximately 26 percent over the unpressurized case; nitrogen had a corresponding increase of 16 percent.

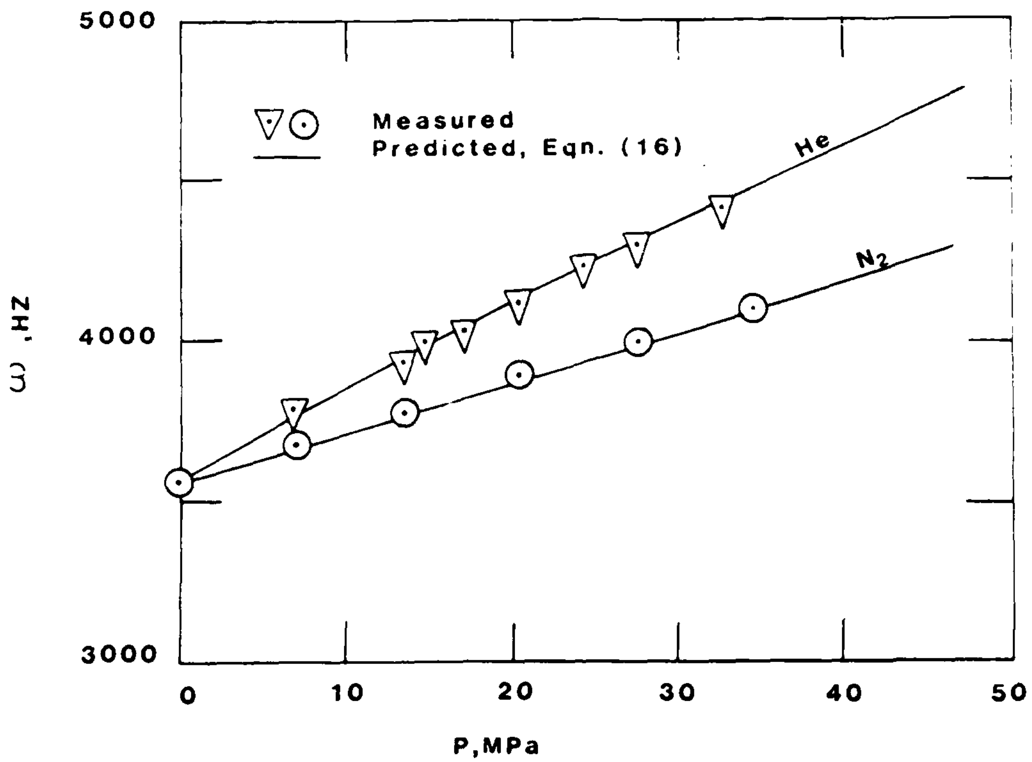


Figure 4. Variation of frequency pressure.

Also shown in Figure 4 is the variation in vibration frequency with pressure as predicted by Equation (23), using the measured zero-pressure frequency ω_0. Helium and nitrogen densities were calculated with a Van der Waals principle of corresponding states [6]. Agreement with the measurements is excellent, differing by less than 4 percent of the frequency increases experienced at the higher pressures and well within the estimated uncertainty of the measurements. Note that the frequency predicted by Equation (23) is somewhat sensitive to the thickness of the bottle δ; for an uncertainty of 0.025 mm the predicted frequencies could have a variation of $\pm$ 30 Hz at the higher pressures.

The agreement between the measurements and Equation (23) might seem somewhat fortuitous considering the rather restrictive simplifications made in the analysis. However, the more comprehensive analysis of Arnold and Warburton [1] indicates that for axial-wave factors $\lambda < 0.60$, thickness ratios $\delta/R >$ 0.03, and $n \geq 2$, the strain energy in an unpressurized cylinder is almost entirely composed of bending, and that the lowest frequency of vibration (of three possible) is comprised mainly of radial motion; these assumptions were made at the outset of this analysis, which should not be extrapolated beyond the limits quoted.

The conditions assumed in the analysis above differ from the tests performed in one important aspect, the end conditions imposed on the cylinder. The displacements of Equation (1) are representative of the condition of "freely-supported" ends, since the test cylinder had hemispherical end caps, a condition somewhere between freely-supported and fixed ends. In their experiments on cylinders with solid and flanged ends, Arnold and Warburton [2] proposed a correlation based on appropriately modified axial-wave factors. Their correlation yields an accurate zero-pressure frequency if the hemispheric-end capped cylinder is considered as solid with end thickness equal to cylinder thickness and with length equal to the cylinder length plus end caps. In this manner, ω_0 is estimated at 3560 Hz, which is remarkably close to the measured value of 3550 Hz.

Equation (23) indicates that for circumferential waves $n > 2$, the effect of pressurization on frequency should be small except at very high pressures. The notion that internal fluid pressure in a container does not affect the frequency [2], although a good guide for most applications, is not precisely correct and could lead to significant underestimations of natural frequencies of high pressure storage vessels under appropriate conditions.

The original impetus for this study came from a need to nondestructively find the fluid pressures inside small, high pressure, sealed containers such as those used on missiles. A quick and simple method is needed to identify those containers which have lost their charge through leakage without time consuming and costly disassembly of the system. This study gives such a method for containers similar to those shown in Figure 2. If natural frequency measurements could be made on the containers, pressures could be easily estimated to an accuracy of ≈ 0.7 MPa, sufficient to determine the operational status of the containers.

V. SUMMARY

This study is summarized as follows:

1. Internal pressurization of thin cylinders can significantly increase vibration frequencies in those circumstances where the strain energy in the unpressurized cylinder is mainly bending. The effect is diminished as the number of circumferential waves is increased and is enhanced with decreasing fluid density.

2. Natural frequencies of vibration of pressurized thin cylinders with $\delta/R > 0.03$ and $\lambda = m\pi R/L < 0.60$ are well correlated by Equation (23) over a range of pressures from 0.34 MPa, for helium and nitrogen gases.

3. Internal pressures in thin cylinders with $\delta/R > 0.03$ can be determined via frequency measurements ("ping test") to an accuracy of approximately 2 percent at pressures up to 34 MPa.

REFERENCES

1. Arnold, R. N. and Warburton, G. B., "Flexural Vibrations of the Walls of Thin Cylindrical Shells Having Freely Supported Ends," Proc. Royal Soc., Series A, v. 197, 1949, pp. 238-256.

2. Arnold, R. N. and Warburton, G. B., "The Flexural Vibrations of Thin Cylinders," Proc. Inst. Mech. Engrs., v. 167, 1953, pp. 62-74.

3. Yu, Yi-Yuan, "Free Vibrations of Thin Cylindrical Shells Having Finite Lengths With Freely Supported and Clamped Edges," Journal of Applied Mechs., Trans. ASME, Vol. 22, No. 4, Dec. 1955, pp. 547-552.

4. Poole, H. A., Jr., "A Stored Gas Actuator Unit for the TOW Missile," Proc. Aerospace Fluid Power Systems and Equipment Conference, pp. 361-376, SAE, 1965.

5. Love, A. E. H., Mathematical Theory of Elasticity, Dover Publications, 1944.

6. Hatsopoulos, G. N. and Keenan, J. H., Principles of General Thermodynamics, John Wiley and Sons, 1965, pp. 242-248.

GLOSSARY

Term	Nomenclature
e_r, e_θ, e_z	Radial, circumferential, and axial strain
E	Young's modulus, 2.07×10^5 MPa
I_n	Modified Bessel function, first kind, n^{th} order
$\Delta \ell_\theta$	Circumferential extension of the middle surface
$\Delta \ell_z$	Axial extension of the middle surface
L	Length of cylinder
m	Number of axial half-waves
n	Number of circumferential waves
p	Pressure
r, θ, z	Radial, circumferential, and axial coordinates
R	Radius of middle surface
t	Time
T	Kinetic energy; $(\ )_c$ and $(\ )_f$ denote cylinder and fluid, respectively
u, v, w	Radial, circumferential, and axial displacements of middle surface
U	Amplitude of radial displacement
V	Potential energy; $(\ )_B$, $(\ )_S$, $(\ )_E$ denote bending, *stretching, and expansion*, respectively
x	Radial distance from middle surface
α	General parameter
δ	Cylinder thickness
$\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{zr}	Shear strains in $r-\theta$, $\theta-z$, and $z-r$ planes, respectively
ν	Distance normal to surface
$\underset{\sim}{\nu}$	Denotes a unit normal
λ	Axial wave factor, $m\pi R/L$
$\Gamma(\alpha)$	Gamma function
ω	Frequency; $(\ )_o$ refers to zero-pressure case
ϕ	Velocity potential of fluid in cylinder
ρ	Density of cylinder
ρ_f	Fluid density
ρ_e	Equivalent density defined by Eqn. (16)
σ	Poisson's ratio, 0.286

Superscripts

$(\dot{\ })$	Denotes differentiation with respect to time
$(\ )'$	Denotes ordinary derivative

APPENDIX

KINETIC ENERGY OF FLUID IN A VIBRATING CIRCULAR CYLINDER

The flow field interior to the cylinder is assumed to be irrotational and to behave incompressibly, since only small pressure variations are anticipated with the small amplitude vibrations considered here. The kinetic energy of the fluid is given in terms of the velocity potential ϕ as

$$T_f = \frac{1}{2}\rho_f \iint_A \phi \frac{\partial\phi}{\partial\nu}\,dA = \frac{1}{2}\rho_f \int_0^L \int_0^{2\pi} \phi \frac{\partial\phi}{\partial\nu}\,r\,d\theta dz \tag{A-1}$$

With the approximation that the unit normal $\underset{\sim}{\nu}$ to the surface is essentially radial, for small displacements, $\partial\phi/\partial\nu \simeq \partial\phi/\partial r$ and, for thin cylinders with $\delta \ll R$, Equation (A-1) can be approximated by

$$T_f \simeq \frac{1}{2}\rho_f \int_0^L \int_0^{2\pi} \phi \frac{\partial\phi}{\partial r}\,R\,d\theta dz \quad . \tag{A-2}$$

The integrand Equation (A-2) is evaluated at $r = R$, the middle surface of the cylinder.

The function

$$\phi = \frac{-\dot{U}L}{m\pi} \sin(m\pi z/L)\cos(n\theta)\frac{I_n(m\pi r/L)}{I_n'(m\pi R/L)} \tag{A-3}$$

satisfies Laplace's equation in the cylindrical region and the boundary condition

$$-\frac{\partial\phi}{\partial r}(R,\theta,z) = \dot{U}\sin(m\pi z/L)\cos(n\theta) \tag{A-4}$$

at the surface $r = R$, in agreement with the displacement given in Equation (1). Further, the net volume flow rate past the surfaces $z = 0$ and $z = L$,

$$q = \int_0 \int_0^{2\pi} \frac{\partial\phi}{\partial z}\,r\,d\theta dr \tag{A-5}$$

is zero; thus, the velocity potential of Equation (A-3) represents the velocity field within a cylinder capped by fixed ends (of any shape).

Substituting Equation (A-3) into Equation (A-1), results in expression

$$T_f = \frac{\pi}{4}\,\frac{(\dot{U})^2 L^3}{m\pi}\,\frac{I_n(m\pi R/L)}{I_n'(m\pi R/L)} \tag{A-6}$$

Equation (A-6) can be simplified with the recurrence relation

$$I'_n(\alpha) = I_{n-1}(\alpha) - \frac{n}{x} I_n(\alpha) \tag{A-7}$$

and, for small arguments, the expression

$$I_n(\alpha) \simeq \frac{(\alpha/2)^n}{\Gamma(n+1)} \tag{A-8}$$

to the form,

$$\frac{I_n(m\pi R/L)}{I'_n(m\pi R/L)} \simeq \frac{1}{\frac{2L}{m\pi R}\,\frac{\Gamma\,(n+1)}{\Gamma\,(n)} - \frac{nL}{m\pi R}} = \frac{m\pi R}{nL} \tag{A-9}$$

Equation (A-6) can thus be written as

$$T_f = \pi\,\rho_f R L^2 (\dot{U})^2/4n \tag{A-10}$$

It is interesting to note that Equation (A-10) is identical to the results obtained by summing the kinetic energies of small axial slices of the cylindrical region, ignoring completely the z-component of velocity in each.

DISTRIBUTION

	No. Copies
A. R. Barbin M. E. Department Auburn University Auburn, AL 36840	2
Dean Houck Naval Air System Command ATTN: AIR 5162C8 Washington, DC 20361	1
Tor Jansen Naval Air Development Center CODE 60134 Warminster, PA 18974	1
Harry Diamond Laboratories ATTN: AELHD-RT-CS, James W. Joyce	1
Richard N. Gottron 2800 Powder Mill Road Adelphi, MD 20783	1
David Keyser Naval Air Development Center CODE 60134 Warminster, PA 18974	1
R. L. McGiboney Naval Air Development Center CODE 60134 Warminster, PA 18974	1
Albertus E. Schmidlin US Army ARRADCOM ATTN: ARDAR-LCN-C, Bldg 65 Dover, NJ 07801	1
Bill Waldon Naval Weapons Center China Lake, CA 93555	1
George W. Fosdick Applied Technology Laboratory ATTN: AAVOL-ATL-ASA Ft. Eustis, VA 23604	1
Commander US Army Training and Doctrine Command Ft. Monroe, VA 23341	1

DISTRIBUTION (Concluded)

	No. Copies
Commander US Army Combined Arms Combat Development Activity Ft. Leavenworth, KS 66027	1
Headquarters Department of the Army ATTN: DAMA-WSM, MAJ Belch Washington, DC 20310	1
Commander US Naval Weapons Center ATTN: J. A. Knecht China Lake, CA 93555	1
US Army Materiel System Analysis Activity ATTN: AMXSY-MP Aberdeen Proving Grounds, MD 21005-5071	1
IIT Research Institute ATTN: GACIAC 10 W. 35th Street Chicago, IL 60616	1
AMSMI-CG	1
AMSMI-RD, Dr. McCorkle	1
Dr. Rhoades	1
-RD-GC, Dr. Yates	1
-RD-GC-C	20
-RD-RE, Dr. Hartman	1
-RD-CS-T	1
-RD-CS-R	15
AMSMI-GC-IP, Mr. Bush	1